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A Hamiltonian formulation for elasticity and thermoelasticity

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Abstract

A Hamiltonian formulation for elasticity and thermoelasticity is proposed and its relation with the corresponding configurational setting is examined. Firstly, a variational principle, concerning the 'inverse motion' mapping, is formulated and the corresponding Euler–Lagrange equations are explored. Next, this Lagrangian formulation is used to define the Hamiltonian density function. The equations of Hamilton are derived in a form which is very similar to the one of the corresponding equations in particle mechanics (finite-dimensional case). From the Hamiltonian formulation it follows that the *canonical momentum* is identified with the pseudomomentum. Furthermore, a meaning for the Poisson bracket is defined and the entailed relations with the canonical variables as well as the balance laws are examined.

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1. Introduction

In a recent paper (Maugin and Kalpakides 2002) we explored the thermoelasticity of Green and Naghdi (1993) in the framework of configurational mechanics, using a Lagrangian formulation. According to this work, the two main concepts of configurational mechanics, the Eshelby stress tensor and the pseudomomentum, are inserted into the picture through invariant arguments for the Lagrangian. In earlier works (Maugin 1993), it was remarked that the equations in the configurational setting could be derived directly from the Lagrangian. This is intimately related to some kind of Hamiltonian structure of elasticity. The object of this paper is to study this relation for the thermoelasticity of Green and Naghdi, which admits a Lagrangian formulation.

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The main point in this approach is that the Lagrangian as well as the Hamiltonian density function should be defined on the *current configuration* instead of the *reference configuration*, which is used in Lagrangian formulations in nonlinear mechanics of solids.

According to the proposed Hamiltonian structure, it is proved that one of the Hamilton equations is the pseudomomentum equation. Note that in the case of particle mechanics (finite-dimensional case), the corresponding Hamilton equation provides the momentum equation. This result justifies the term 'canonical momentum', proposed by Maugin (1993), for the 'pseudomomentum' density function.

Furthermore, having obtained a Hamiltonian structure, the concept of Poisson bracket and its relation with the conservation (or non-conservation) laws are studied.

Generally, if not otherwise denoted, all indices will range from 1 to 3. Two distinct differential operators, $\partial/\partial X_L$ and $\mathrm{d}/\mathrm{d}X_L$, are used; the former is the usual partial derivative operator while the latter denotes the partial derivative which accounts for the underlying function composition. For instance,

$$\frac{\mathrm{d}}{\mathrm{d}X_L} F(X_B, x_i(X_B)) = \frac{\partial F}{\partial X_L} + \frac{\partial F}{\partial x_i}\frac{\partial x_i}{\partial X_L}.$$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}x_i} \Phi(x_j, X_L(x_j)) = \frac{\partial \Phi}{\partial x_i} + \frac{\partial \Phi}{\partial X_L}\frac{\partial X_L}{\partial x_i}.$$

Besides, the usual notation Grad $F = \nabla F = \mathrm{d}F/\mathrm{d}X_L$, Div $\mathbf{F} = \mathrm{d}F_L/\mathrm{d}X_L$ and $\dot{F} = \mathrm{D}F/\mathrm{D}t$ for gradient, divergence and material time derivative, respectively, are used. The gradient and the divergence operator for a function $\mathbf{f}$ (or a scalar function f) defined on $\mathbf{x}$ will be denoted by grad $f = \mathrm{d}f/\mathrm{d}x_i$ and div $\mathbf{f} = \mathrm{d}f_i/\mathrm{d}x_i$, respectively.

2. Preliminary results

The motion of a thermoelastic body is described by the smooth mappings

$$\mathbf{x} = \chi(\mathbf{X}, t) \qquad \alpha = \alpha(\mathbf{X}, t) \tag{2.1}$$

where $\alpha = \alpha(\mathbf{X}, t)$ is the *thermal displacement field*, $\mathbf{X}$ is the material space variable and $\mathbf{x}$ is the spatial position of the particle $\mathbf{X}$ at time t. In a coordinate system, these variables will be written as X_L, $L = 1, 2, 3$, and x_i, $i = 1, 2, 3$, respectively. The temperature field is defined to be the time derivative of α, thus

$$\theta(\mathbf{X}, t) := \frac{\partial}{\partial t}\alpha(\mathbf{X}, t).$$

Next, we give some very fundamental notions and results of the Green and Naghdi thermoelasticity (Green and Naghdi 1993). The field equations, i.e. the momentum and energy equations, are given, respectively, as follows:

$$\frac{\partial(\rho\mathbf{v})}{\partial t} - \mathrm{Div}\,\mathbf{T} = \mathbf{0} \tag{2.2}$$

$$-\left(\frac{\mathrm{D}\Psi}{\mathrm{D}t} + \frac{\mathrm{D}\theta}{\mathrm{D}t}\eta\right) + \mathrm{tr}(\mathbf{T}\dot{\mathbf{F}}) - \mathbf{S}\cdot\nabla\theta = 0 \tag{2.3}$$

where ρ is the mass density in the reference configuration, $\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}$ is the velocity field, Ψ is the free energy function per unit undeformed volume, $\mathbf{T}$ is the first Piola–Kirchhoff stress, $\mathbf{F}$ is the deformation gradient, $\mathbf{S}$ is the entropy flux vector and η is the entropy density per unit undeformed volume.

The constitutive equations are of the form

$$\mathbf{T} = \frac{\partial \Psi}{\partial \mathbf{F}} \qquad \mathbf{S} = -\frac{\partial \Psi}{\partial \boldsymbol{\beta}} \qquad \eta = -\frac{\partial \Psi}{\partial \dot{\alpha}} \qquad \Psi = \Psi(\mathbf{X}, \mathbf{F}, \boldsymbol{\beta}, \dot{\alpha}) \tag{2.4}$$

where $\boldsymbol{\beta} = \nabla\alpha$. It is worth noting that the Green and Naghdi theory is an alternative formulation of what is called hyperbolic thermoelasticity in which thermal disturbances propagate with finite wave speed. That is why it does admit a variational formulation (Maugin and Kalpakides 2002). Nevertheless, the interesting point of this formulation is that the classical dissipative thermoelasticity can be provided, by a proper choice of constitutive assumptions, as well. Consequently, one could generalize the proposed Hamiltonian formulation to cover dissipative thermoelasticity too.

It seems that the natural setting for the Hamiltonian formulation of elasticity is the current configuration B_t, i.e. the space of placements (we denote by B_r the reference configuration). Hence, in this work all quantities are defined in this space. This, in turn, leads us to use the inverse motion mapping $\mathbf{X} = \mathcal{X}(\mathbf{x}, t)$ instead of the motion mapping $\mathbf{x} = \chi(\mathbf{X}, t)$, which hereafter will be called the direct motion mapping. One can easily show that the following relation between the variables in reference and current configuration holds,

$$\frac{\partial x_i}{\partial t} = -\frac{\partial x_i}{\partial X_L}\frac{\partial X_L}{\partial t} \qquad \text{or} \qquad \mathbf{v} = -(\nabla\chi)\mathbf{V} \tag{2.5}$$

where $\mathbf{V}$ is the 'velocity' of the inverse motion.

Now, we would like to draw the attention of the reader to the differential operator with respect to t. Apart from the usual material derivative, we will need an additional differential operator with respect to t, while we consider the inverse motion mapping. In that case, note that x_i and t make up the space of independent variables, thus, the partial time differentiation is carrying out for fixed x. Consequently, we define the time differentiation operator $\mathrm{d}/\mathrm{d}t$ on any function $f(\mathbf{X}, t)$ by the following meaning:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial t}. \tag{2.6}$$

Certainly, for a function ϕ defined directly on $\mathbf{x}$ and t ($\phi = \phi(\mathbf{x}, t)$) this operator provides the simple partial differentiation, i.e.

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial \phi}{\partial t}. \tag{2.7}$$

Furthermore, one can prove the following formulae concerning the differentiation of J_F, Grad χ and grad $\mathcal{X}$,

$$\begin{aligned} \frac{\partial x_{i,K}}{\partial X_{L,j}} &= -x_{i,L}x_{j,K} \qquad & \frac{\partial X_{K,i}}{\partial x_{j,L}} &= -X_{L,i}X_{K,j} \\ \frac{\partial J_F}{\partial x_{i,L}} &= J_F X_{L,i} \qquad & \frac{\partial J_F}{\partial X_{L,i}} &= -J_F x_{i,L} \end{aligned} \tag{2.8}$$

where J_F is the determinant of $\mathbf{F}$. Note that $J_{F^{-1}} = J_F^{-1}$.

Also, the following relation[2] can be proved:

$$\frac{\mathrm{d}J_{F^{-1}}}{\mathrm{d}t} = J_{F^{-1}} \,\mathrm{Div}\, \mathbf{V}. \tag{2.9}$$

Using equation (2.6) and the identity (Maugin 1993)

$$\frac{\mathrm{d}}{\mathrm{d}X_L}(J_F X_{L,i}) \equiv 0$$

[2] Note the similarity with the relation $\frac{\mathrm{D}J_F}{\mathrm{D}t} = J_F \,\mathrm{div}\, \mathbf{v}$ which can be found in any standard text in continuum mechanics.

we can obtain the following relation that is useful in forthcoming calculations (Podio-Guidugli 2001),

$$\operatorname{div}\mathbf{L} = J_F{}^{-1}\operatorname{Div}(\mathbf{L}J_F(\mathbf{F}^{-1})^T)$$

or in index notation

$$\frac{\mathrm{d}L_{ij}}{\mathrm{d}x_j} = J_F{}^{-1}\frac{\mathrm{d}}{\mathrm{d}X_M}(L_{ij}J_F X_{M,j}) \tag{2.10}$$

where L_{ij} is any spatial Cartesian tensor.

All quantities involved in our problem, will appear both as functions of $\mathbf{X}, t$ (material configuration) or of $\mathbf{x}, t$ (current configuration). The thermal displacement function in the current configuration will be denoted as $\bar{\alpha}(\mathbf{x}, t)$. It is related to the corresponding quantity in reference configuration, i.e. equation $(2.1)_2$, through the relations

$$\alpha = \bar{\alpha}\circ\chi \qquad \bar{\alpha} = \alpha\circ\mathcal{X}$$

where $\circ$ denotes the composition between functions. Therefore, their derivatives will fulfil the following relations:

$$\frac{\partial\alpha}{\partial X_L} = \frac{\partial\bar{\alpha}}{\partial x_i}x_{i,L} \qquad \frac{\partial\alpha}{\partial t} = \frac{\partial\bar{\alpha}}{\partial t} - \frac{\partial\bar{\alpha}}{\partial x_i}x_{i,A}\frac{\partial X_A}{\partial t}. \tag{2.11}$$

The free energy function in the reference configuration and in the current configuration, respectively, will be of the form

$$\Psi = \Psi(\mathbf{X}, \operatorname{Grad}\chi, \operatorname{Grad}\alpha, \dot{\alpha}) \qquad \bar{\Psi} = \bar{\Psi}(\mathcal{X}(x,t), \operatorname{grad}\mathcal{X}, \operatorname{grad}\bar{\alpha}, \dot{\bar{\alpha}}). \tag{2.12}$$

The two functions are linked through the relation

$$\bar{\Psi} = J_F{}^{-1}\Psi\circ\mathcal{X}. \tag{2.13}$$

3. The Lagrangian formulation

We start with the following definition of the Lagrange density function:

Definition. *The Lagrangian function of a thermoelastic body, defined in the current configuration, has the following form,*

$$\begin{aligned}\Lambda\left(\mathcal{X}_L, \frac{\partial\mathcal{X}_L}{\partial t}, \frac{\partial\bar{\alpha}}{\partial t}, \frac{\partial\mathcal{X}_L}{\partial x_i}, \frac{\partial\bar{\alpha}}{\partial x_i}\right) &= \bar{K} - \bar{\Psi} \\ &= J_F{}^{-1}\frac{1}{2}\bar{\varrho}(x_i)\frac{\partial\mathcal{X}_K}{\partial t}C_{KL}\frac{\partial\mathcal{X}_L}{\partial t} - \bar{\Psi}\left(\mathcal{X}_L, \frac{\partial\mathcal{X}_L}{\partial x_i}, \frac{\partial\bar{\alpha}}{\partial x_i}, \frac{\partial\bar{\alpha}}{\partial t}\right)\end{aligned} \tag{3.1}$$

where $\bar{\rho}$ is the mass density in the current configuration. Note that $\bar{K}$ is nothing else but the kinetic energy function defined in the current configuration, i.e. $\bar{K} = J_F{}^{-1}K\circ\mathcal{X}$, consequently, according to the above definition we have

$$\Lambda = J_F{}^{-1}L\circ\mathcal{X}$$

where L denotes the Lagrangian density function defined per unit volume in the reference configuration.

Consider now a time interval $[t_1, t_2]$, then the corresponding action functional will be of the form

$$\mathcal{L}[\mathcal{X}, \bar{\alpha}] = \int_{t_1}^{t_2}\int_{B_t}\Lambda\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2}$$

Note the relation with the standard energy functional used in solid mechanics:

$$\int_{t_1}^{t_2}\int_{B_t} \Lambda \,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{B_r} J_F{}^{-1} L\,\mathrm{d}X\,\mathrm{d}t.$$

Furthermore we assume boundary and initial conditions of the following form,

$$X_L(x_i,t) = f_L(x_i,t) \qquad \text{for all } x_i \in \partial B_t \text{ and for all } t \in [t_1,t_2] \tag{3.3}$$

$$X_L(x_i,t_1) = f_L^1(x_i) \qquad \text{for all } x_i \in B_{t_1} \qquad X_L(x_i,t_2) = f_L^2(x_i) \qquad \text{for all } x_i \in B_{t_2} \tag{3.4}$$

and

$$\bar{\alpha}(x_i,t) = g(x_i,t) \qquad \text{for all } x_i \in \partial B_t \text{ and for all } t \in [t_1,t_2] \tag{3.5}$$

$$\bar{\alpha}(x_i,t_1) = g^1(x_i) \qquad \text{for all } x_i \in B_{t_1} \qquad \bar{\alpha}(x_i,t_2) = g^2(x_i) \qquad \text{for all } x_i \in B_{t_2} \tag{3.6}$$

where f_L, f_L^1, f_L^2, g, g^1 and g^2 are given functions. Then, requiring an extremum for the functional $\mathcal{L}$, we obtain the Euler–Lagrange equations:

$$\frac{\partial\Lambda}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial X_{L,i}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial X_{L,t}}\right) = 0 \tag{3.7}$$

$$\frac{\partial\Lambda}{\partial\bar{\alpha}} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial\bar{\alpha}_{,i}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial\bar{\alpha}_{,t}}\right) = 0 \tag{3.8}$$

for all x in B_r and for all t in the interval $[t_1, t_2]$.

The boundary conditions (3.3) and (3.5) correspond to what are usually called essential boundary conditions (i.e. X_L and $\bar{\alpha}$ are prescribed on the boundary ∂B_t). Using the standard procedure of the calculus of variations, one can obtain the Euler–Lagrange equations (3.7) and (3.8) for the case of natural boundary conditions as well.

Using relations (2.14) and (2.4) we carry out the following calculations for the determination of the intermediate terms of equations (3.7) and (3.8):

$$\begin{aligned}
\frac{\partial\bar{\Psi}}{\partial X_{L,i}} &= \frac{\partial J_F{}^{-1}}{\partial X_{L,i}}\Psi + J_F{}^{-1}\left[\frac{\partial\Psi}{\partial x_{j,A}}\frac{\partial x_{j,A}}{\partial X_{L,i}} + \frac{\partial\Psi}{\partial\left(\frac{\partial\alpha}{\partial X_A}\right)}\frac{\partial\left(\frac{\partial\alpha}{\partial X_A}\right)}{\partial X_{L,i}} + \frac{\partial\Psi}{\partial\left(\frac{\partial\alpha}{\partial t}\right)}\frac{\partial\left(\frac{\partial\alpha}{\partial t}\right)}{\partial X_{L,i}}\right] \\
&= J_F{}^{-1}x_{i,L}\Psi + J_F{}^{-1}\left[-T_{Aj}x_{j,L}x_{i,A} + S_A\frac{\partial\bar{\alpha}}{\partial x_j}x_{j,L}x_{i,A} - \eta\frac{\partial\bar{\alpha}}{\partial x_j}x_{j,L}x_{i,A}\right] \\
&= J_F{}^{-1}x_{i,L}\Psi - J_F{}^{-1}\left[T_{Aj}x_{j,L}x_{i,A} - S_A\beta_L x_{i,A} + \eta\beta_L\frac{\partial x_i}{\partial t}\right].
\end{aligned} \tag{3.9}$$

Similarly, we obtain

$$\frac{\partial\bar{\Psi}}{\partial\left(\frac{\partial X_L}{\partial t}\right)} = J_F{}^{-1}\eta\beta_L \tag{3.10}$$

$$\frac{\partial\bar{K}}{\partial X_{L,i}} = J_F{}^{-1}x_{i,L}K - J_F{}^{-1}\rho x_{j,L}\frac{\partial x_j}{\partial t}\frac{\partial x_i}{\partial t} \tag{3.11}$$

$$\frac{\partial\bar{K}}{\partial\left(\frac{\partial X_L}{\partial t}\right)} = -J_F{}^{-1}\rho\frac{\partial x_i}{\partial t}x_{i,L}. \tag{3.12}$$

Consequently, the partial derivatives of the Lagrangian Λ are given as follows:

$$\begin{aligned}
\frac{\partial\Lambda}{\partial X_{L,i}} &= \frac{\partial\bar{K}}{\partial X_{L,i}} - \frac{\partial\bar{\Psi}}{\partial X_{L,i}} \\
&= J_F{}^{-1}\left[x_{i,A}(\delta_{AL}(K-\Psi) + T_{Aj}x_{j,L} - S_A\beta_L) - \frac{\partial x_i}{\partial t}\left(\rho\frac{\partial x_j}{\partial t}x_{j,L} - \eta\beta_L\right)\right]
\end{aligned}$$

or

$$\frac{\partial \Lambda}{\partial X_{L,i}} = J_F{}^{-1}\left(x_{i,A}b_{AL} + \frac{\partial x_i}{\partial t}P_L\right) \tag{3.13}$$

where

$$b_{AL} = -(L\delta_{AL} + T_{Aj}x_{j,L} - S_A\beta_L) \tag{3.14}$$

$$P_L = -\left(\rho\frac{\partial x_j}{\partial t}x_{j,L} + \eta\beta_L\right) \tag{3.15}$$

are the *Eshelby stress tensor* and *pseudomomentum* or *canonical momentum*, respectively.

Similarly, the partial derivative of Λ with respect to the variable $\frac{\partial X_L}{\partial t}$ is

$$\frac{\partial \Lambda}{\partial\left(\frac{\partial X_L}{\partial t}\right)} = -J_F{}^{-1}\left(\rho\frac{\partial x_i}{\partial t}x_{i,L} - \eta\beta_L\right) = J_F{}^{-1}P_L. \tag{3.16}$$

What remains is to differentiate expressions (3.13) and (3.16) with respect to x_i and t. We start differentiating with respect to x_i with the aid of equation (2.10):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \Lambda}{\partial X_{L,i}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x_i}\left[J_F{}^{-1}\left(-x_{i,A}b_{AL} + \frac{\partial x_i}{\partial t}P_L\right)\right]\\ &= \frac{1}{J_F}\frac{\mathrm{d}}{\mathrm{d}X_M}\left[J_F{}^{-1}\left(-x_{i,A}b_{AL} + \frac{\partial x_i}{\partial t}P_L\right)J_F X_{M,i}\right]\\ &= -J_F{}^{-1}\frac{\mathrm{d}}{\mathrm{d}X_M}(b_{ML} + V_M P_L).\end{aligned} \tag{3.17}$$

Next, we proceed to the differentiation with respect to t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Lambda}{\partial\left(\frac{\partial X_L}{\partial t}\right)}\right) = \frac{\mathrm{d}J_F{}^{-1}}{\mathrm{d}t}P_L + J_F{}^{-1}\frac{\mathrm{d}P_L}{\mathrm{d}t}.$$

Evoking now equations (2.6) and (2.9), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Lambda}{\partial\left(\frac{\partial X_L}{\partial t}\right)}\right) &= J_F{}^{-1}\left(\frac{\mathrm{d}V_M}{\mathrm{d}X_M}\right)P_L + J_F{}^{-1}\left(\frac{\partial P_L}{\partial t} + \frac{\partial P_L}{\partial X_M}\frac{\partial X_M}{\partial t}\right)\\ &= J_F{}^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}X_M}(P_L V_M) + \frac{\partial P_L}{\partial t}\right).\end{aligned} \tag{3.18}$$

Note that, as P_L was defined by equation (3.15), it is a function of $\mathbf{X}$ and t, thus $\partial P_L/\partial X_L = \mathrm{d}P_L/\mathrm{d}X_L$.

Finally, we calculate the first term of the Euler–Lagrange equation (3.7):

$$\frac{\partial \Lambda}{\partial X_L} = \frac{\partial}{\partial X_L}(J_F{}^{-1}L) = J_F{}^{-1}\frac{\partial L}{\partial X_L}. \tag{3.19}$$

Thus, equation (3.7) takes on the form

$$J_F{}^{-1}\left[\frac{\partial L}{\partial X_L} + \frac{\mathrm{d}}{\mathrm{d}X_M}(b_{ML} + V_M P_L) - \frac{\mathrm{d}}{\mathrm{d}X_M}(P_L V_M) - \frac{\partial P_L}{\partial t}\right] = 0$$

or

$$\frac{\partial L}{\partial X_L} + \frac{\mathrm{d}b_{ML}}{\mathrm{d}X_M} - \frac{\partial P_L}{\partial t} = 0. \tag{3.20}$$

Equation (3.20) is the pseudomomentum equation for thermoelasticity (see Maugin and Kalpakides (2002)). The interesting point here lies in the way the equation has been obtained; it is proved that the Euler–Lagrange equations for the inverse motion mapping leads directly to the equation of material momentum. Let us proceed to the second of the Euler–Lagrange equations, i.e. equation (3.8), calculating one by one all of its terms,

$$\frac{\partial\bar{\Psi}}{\partial\left(\frac{\partial\bar{\alpha}}{\partial x_i}\right)} = J_F{}^{-1}\left[\frac{\partial\Psi}{\partial\left(\frac{\partial\alpha}{\partial X_L}\right)}\frac{\partial\left(\frac{\partial\alpha}{\partial X_L}\right)}{\partial\left(\frac{\partial\bar{\alpha}}{\partial x_i}\right)} + \frac{\partial\Psi}{\partial\left(\frac{\partial\alpha}{\partial t}\right)}\frac{\partial\left(\frac{\partial\alpha}{\partial t}\right)}{\partial\left(\frac{\partial\bar{\alpha}}{\partial x_i}\right)}\right]$$

$$= J_F{}^{-1}\left(-S_L x_{i,L} + \eta x_{i,L}\frac{\partial X_L}{\partial t}\right) = J_F{}^{-1}\left(-S_L x_{i,L} - \eta\frac{\partial x_i}{\partial t}\right) \tag{3.21}$$

$$\frac{\partial\bar{\Psi}}{\partial\left(\frac{\partial\bar{\alpha}}{\partial t}\right)} = J_F{}^{-1}\frac{\partial\Psi}{\partial\left(\frac{\partial\alpha}{\partial t}\right)}\frac{\partial\left(\frac{\partial\alpha}{\partial t}\right)}{\partial\left(\frac{\partial\bar{\alpha}}{\partial t}\right)} = J_F{}^{-1}\eta. \tag{3.22}$$

Hence, we easily obtain with the aid of equations (2.9) and (2.11)

$$\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial\left(\frac{\partial\bar{\alpha}}{\partial x_i}\right)}\right) = \frac{\mathrm{d}}{\mathrm{d}x_i}\left[J_F{}^{-1}\left(-S_L x_{i,L} - \eta\frac{\partial x_i}{\partial t}\right)\right] = J_F{}^{-1}\frac{\mathrm{d}}{\mathrm{d}X_M}(S_M - \eta V_M) \tag{3.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\bar{\Psi}}{\partial\left(\frac{\partial\bar{\alpha}}{\partial t}\right)}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(J_F{}^{-1}\eta) = J_F{}^{-1}\frac{\mathrm{d}V_M}{\mathrm{d}X_M}\eta + J_F{}^{-1}\frac{\mathrm{d}\eta}{\mathrm{d}t}. \tag{3.24}$$

Inserting equations (3.23) and (3.24) into equation (3.8), we finally obtain

$$\frac{\mathrm{d}S_M}{\mathrm{d}X_M} + \frac{\mathrm{d}\eta}{\mathrm{d}t} = 0 \tag{3.25}$$

which is nothing else but the entropy equation (Green and Naghdi 1993, Maugin and Kalpakides 2002). It should be mentioned that equations (3.20) and (3.25) are not only of theoretical importance. It has been proved (Dascalu and Maugin 1995) that these equations can provide path-domain independent expression for the thermoelastic energy-release rate which is of practical importance in thermoelastic fracture problems.

Remark. It is easily understood that the above analysis covers completely the case of elasticity as well. In this case, it is enough to remark that the Lagrangian function has fewer arguments and it is of the form

$$\Lambda\left(\mathcal{X}_L, \frac{\partial\mathcal{X}_L}{\partial t}, \mathcal{X}_{L,i}\right) = J_F{}^{-1}\frac{1}{2}\bar{\varrho}(x_i)\frac{\partial\mathcal{X}_K}{\partial t}C_{KL}\frac{\partial\mathcal{X}_L}{\partial t} - \bar{\Psi}(\mathcal{X}_L, \mathcal{X}_{L,i}). \tag{3.26}$$

Also, one takes only one Euler–Lagrange equation in the same form as equation (3.20), but with reduced definitions for Eshelby stress tensor b_{ML} and pseudomomentum P_L given now by the equations

$$b_{AL} = -(L\delta_{AL} + T_{Aj}x_{j,L}) \qquad P_L = -\rho\frac{\partial x_j}{\partial t}x_{j,L}. \tag{3.27}$$

This result is in accordance with Maugin (1993), where the original idea of a Lagrangian formulation of a pseudomomentum equation, using the inverse motion, appears for the first time.

4. The Hamiltonian formulation

In this section, we will show that equations (3.20) and (3.25) admit a Hamiltonian formulation, hence it is justified to call them *canonical equations* (Maugin 1993). In accordance with the Hamiltonian mechanics (Goldstein 1950, Rund 1966), we define as canonical momenta the quantities

$$\pi_L = \frac{\partial \Lambda}{\partial \left(\frac{\partial X_L}{\partial t}\right)} \qquad \bar{\eta} = \frac{\partial \Lambda}{\partial \left(\frac{\partial \bar{\alpha}}{\partial t}\right)}. \tag{4.1}$$

Note that π_L and $\bar{\eta}$ are defined on B_t; they are related to the pseudomomentum and entropy functions through the relations

$$\pi_L = J_F{}^{-1} P_L \circ \mathcal{X} \qquad \bar{\eta} = J_F{}^{-1} \eta \circ \mathcal{X}.$$

To fix ideas, firstly we are confined in the framework of elasticity.

4.1. The Hamilton equations for elasticity

We start giving a definition for the Hamiltonian function:

Definition. *The Hamiltonian function of an elastic body is defined to be*

$$H(X_L, \pi_L) = \pi_L \frac{\partial X_L}{\partial t} - \Lambda \left(X_L, \frac{\partial X_L}{\partial t}, \frac{\partial X_L}{\partial x_i} \right). \tag{4.2}$$

Furthermore, we must define the variational derivative for our problem. To this end, let us consider the mapping $\mathcal{H}$ defined by the following relation:

$$\mathcal{H}[X_L, \pi_L] := \int_\Omega H(X_L, \pi_L)\, \mathrm{d}x$$

where Ω is any regular subset of B_t. It is important to note that $\mathcal{H}$ is a mapping which has for domain the space of all smooth functions X_L and P_L (defined on $B_t \times [t_1, t_2]$) and fulfil some boundary conditions on the boundary ∂B_t. The range of $\mathcal{H}$ is in the space of all smooth functions on $[t_1, t_2]$. Let us assume that $\mathcal{H}$ is a Fréchet differentiable mapping with respect to its arguments X_L and P_L. Also, let δX_L be any admissible variation of the inverse motion mapping X_L.

Denoting

$$\mathcal{H}_X(\epsilon) = \mathcal{H}[X_L + \epsilon \delta X_L, \pi_L] \qquad \text{for any sufficiently small real } \epsilon,$$

the Fréchet differential of $\mathcal{H}$ with respect to X_L will be[3]

$$\delta \mathcal{H}[X_L, P_L; \delta X_L] = \lim_{\epsilon \to 0} \frac{\mathcal{H}_X(\epsilon) - \mathcal{H}_X(o)}{\epsilon} \tag{4.3}$$

where the limit on the rhs is in the sense of sup-convergence[4], i.e. uniformly in $[t_1, t_2]$.

Under these assumptions, we can calculate $\delta\mathcal{H}$ as follows:

$$\begin{aligned}
\delta \mathcal{H}[X_L, \pi_L; \delta X_L] &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \left[\int_\Omega H(X_L + \epsilon \delta X_L, \pi_L)\, \mathrm{d}x \right]_{\epsilon=0} \\
&= \int_\Omega \left[\frac{\partial H}{\partial X_L} \delta X_L + \frac{\partial H}{\partial X_{L,t}} (\delta X_L)_{,t} + \frac{\partial H}{\partial X_{L,i}} (\delta X_L)_{,i} \right] \mathrm{d}x \\
&= \int_\Omega \left[-\frac{\partial H}{\partial X_L} \delta X_L + \left(-\frac{\partial \Lambda}{\partial X_{L,t}} + \pi_L \right) (\delta X_L)_{,t} - \frac{\partial H}{\partial X_{L,i}} (\delta X_L)_{,i} \right] \mathrm{d}x
\end{aligned}$$

for all t in $[t_1, t_2]$.

[3] More accurately, this is the weak differential of $\mathcal{H}$ in the direction of $\mathbf{X}$.

[4] Keep in mind that $\mathcal{H}_X(\epsilon)$ is a smooth function (at least C^1) of t.

Evoking now definition $(4.1)_1$ we can write

$$\delta\mathcal{H}[X_L,\pi_L;\delta X_L]=\int_\Omega\left[\frac{\partial H}{\partial X_L}\delta X_L+\frac{\partial H}{\partial X_{L,i}}(\delta X_L)_{,i}\right]\mathrm{d}x+\int_{\partial\Omega}\frac{\partial H}{\partial X_{L,i}}\delta X_L\nu_i\,\mathrm{d}x \tag{4.4}$$

where ν_i is the unit normal vector on $\partial\Omega$.

Remark. In the case where the domain of integration is the whole body B_t, the form of equation (4.4) depends on the boundary data. Assuming essential boundary conditions on ∂B_t, that is

$$X_L(x_i,t)\text{ is prescribed on }\partial B_t,\qquad\text{for all }t\text{ in }[t_1.t_2] \tag{4.5}$$

all the variations δX_L will vanish on ∂B_t, thus equation (4.4) takes the form

$$\delta\mathcal{H}[X_L,\pi_L;\delta X_L]==\int_{B_t}\left[\frac{\partial H}{\partial X_L}-\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial X_{L,i}}\right)\right]\delta X_L\,\mathrm{d}x. \tag{4.6}$$

In the case of natural boundary conditions, we have boundary data of the form

$$\frac{\partial H}{\partial X_{L,i}}\nu_i=F_L\quad\text{on}\quad\partial B_t,\qquad\text{for all }t\text{ in }[t_1.t_2]$$

where F_L is a kind of material force on the boundary.

Then the Hamiltonian mapping $\mathcal{H}$ will take the form

$$\mathcal{H}[X_L,\pi_L]:=\int_{\partial B_t}H(X_L,\pi_L)\,\mathrm{d}x-\int_{\partial B_t}F_LX_L\,\mathrm{d}x$$

and its Fréchet differential with respect to X_L will be[5]

$$\begin{aligned}\delta\mathcal{H}[X_L,\pi_L;\delta X_L]&=\int_{\partial B_t}\left[\frac{\partial H}{\partial X_L}\delta X_L+\frac{\partial H}{\partial X_{L,i}}(\delta X_L)_{,i}\right]\mathrm{d}x+\int_{\partial B_t}\left[\frac{\partial H}{\partial X_{L,i}}\nu_i-F_L\right]\delta X_L\,\mathrm{d}x\\&=\int_{\partial B_t}\left[\frac{\partial H}{\partial X_L}\delta X_L+\frac{\partial H}{\partial X_{L,i}}(\delta X_L)_{,i}\right]\mathrm{d}x.\end{aligned}$$

Consequently, for any case of boundary conditions the Fréchet differential keeps the same form[6].

Next, we define as the *variational derivative of H with respect to* X_L

$$\frac{\delta H}{\delta X_L}=\frac{\partial H}{\partial X_L}-\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial X_{L,i}}\right)\qquad\forall t\in[t_1,t_2]. \tag{4.7}$$

Thus, equation (4.4) can be written as

$$\delta\mathcal{H}[X_L,\pi_L;\delta X_L]=\int_\Omega\frac{\delta H}{\delta X_L}\delta X_L+\int_{\partial\Omega}\frac{\partial H}{\partial X_{L,i}}\delta X_L\nu_i\,\mathrm{d}x.$$

Similarly, the Fréchet differential of $\mathcal{H}$ with respect to π_L will be given by the following relation:

$$\begin{aligned}\delta\mathcal{H}[X_L,\pi_L;\delta\pi_L]&=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\int_\Omega H(X_L,\pi_L+\epsilon\delta\pi_L)\,\mathrm{d}x\right]_{\epsilon=0}\\&=\int_\Omega\frac{\partial H}{\partial\pi_L}\delta\pi_L\,\mathrm{d}x=\int_\Omega\frac{\delta H}{\delta\pi_L}\delta\pi_L\,\mathrm{d}x.\end{aligned} \tag{4.8}$$

[5] Supposing that F_L is a 'dead' material force to secure that it is not affected by the variations of X_L.

[6] One can easily prove the same result for the mixed type boundary conditions, as well.

Thus, the corresponding variational derivative with respect to π_L was defined as

$$\frac{\delta H}{\delta \pi_L} = \frac{\partial H}{\partial \pi_L} \qquad \forall t \in [t_1, t_2]. \tag{4.9}$$

Generally, in virtue of equations (4.7) and (4.9) we can use the operators $\delta/\delta X_L$ and $\delta/\delta\pi_L$ for any smooth function $f = f(X_L, X_{L,i}, \pi_L)$ as follows:

$$\frac{\delta f}{\delta X_L} = \frac{\partial f}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial f}{\partial X_{L,i}}\right) \qquad \frac{\delta f}{\delta \pi_L} = \frac{\partial f}{\partial \pi_L}. \tag{4.10}$$

Now, one can easily prove that the following relations hold:

$$\frac{\delta \pi_L}{\delta X_M} = 0 \qquad \frac{\delta \pi_L}{\delta \pi_M} = \delta_{LM} \qquad \frac{\delta X_L}{\delta X_M} = \delta_{LM}. \tag{4.11}$$

Concluding, we can state that *the Hamilton equations for elasticity are given by the equations*

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}t} = -\frac{\delta H}{\delta X_L} \qquad \frac{\mathrm{d}X_L}{\mathrm{d}t} = \frac{\delta H}{\delta \pi_L} \qquad \forall t \in [t_1, t_2]. \tag{4.12}$$

Using the definition of the variational derivative we can easily prove that the first of the Hamilton equations is nothing other than the pseudomomentum equation for elasticity. Indeed, with the aid of definition (4.1), the lhs of equation $(4.12)_1$ can be written as

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}}\left(\frac{\partial \Lambda}{\partial X_{L,t}}\right).$$

On the other hand, the rhs of equation $(4.12)_1$ by virtue of equations (4.7) and (4.2) is written as

$$-\frac{\delta H}{\delta X_L} = \frac{\partial \Lambda}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \Lambda}{\partial X_{L,i}}\right).$$

Hence, equation $(4.12)_1$ is identified with equation (3.7) which we have already proved provides the pseudomomentum equation for elasticity (see the remark at the end of section 3). The second of the Hamilton equations, i.e. equation $(4.12)_2$, is a simple consequence of definition[7] (4.2).

4.2. *The Hamilton equations for thermoelasticity:*

Following a similar line of development we define the Hamiltonian function for thermoelasticity:

Definition. *The Hamiltonian function of a thermoelastic body is defined to be of the form*

$$H(X_L, \bar{\alpha}, \pi_L, \bar{\eta}) = \pi_L \frac{\partial X_L}{\partial t} + \bar{\eta}\frac{\partial \bar{\alpha}}{\partial t} - \Lambda\left(X_L, \frac{\partial X_L}{\partial t}, \frac{\partial \bar{\alpha}}{\partial t}, \frac{\partial X_L}{\partial x_i}, \frac{\partial \bar{\alpha}}{\partial x_i}\right). \tag{4.13}$$

Note that now we have an additional canonical momentum and an additional generalized coordinate given by $\bar{\eta}$ and $\bar{\alpha}$, respectively. Also, although $\bar{\alpha}$ appears in the argument of H, the latter depends on the derivatives of $\bar{\alpha}$. Furthermore, the mapping $\mathcal{H}$ takes the form

$$\mathcal{H}[X_L, \bar{\alpha}, \pi_L, \bar{\eta}] := \int_\Omega H(X_L, \bar{\alpha}, \pi_L, \bar{\eta})\,\mathrm{d}x.$$

Following the same line of arguments, the variational derivatives of $\mathcal{H}$ are given as

$$\frac{\delta H}{\delta X_L} = \frac{\partial H}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial X_{L,i}}\right) = -\frac{\partial \Lambda}{\partial X_L} + \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial \Lambda}{\partial X_{L,i}}\right) \tag{4.14}$$

[7] Note that according to definition (2.6), $\frac{\partial X_L}{\partial t} = \frac{\mathrm{d}X_L}{\mathrm{d}t}$.

$$\frac{\delta H}{\delta\bar{\alpha}} = -\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial\bar{\alpha}_{,i}}\right) = \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial L}{\partial\bar{\alpha}_{,i}}\right) \tag{4.15}$$

and

$$\frac{\delta H}{\delta\pi_L} = \frac{\partial H}{\partial\pi_L} = \frac{\partial X_L}{\partial t} \qquad \frac{\delta H}{\delta\bar{\eta}} = \frac{\partial H}{\partial\bar{\eta}} = \frac{\partial\bar{\alpha}}{\partial t} \qquad \text{for all } t \text{ in } [t_1, t_2]. \tag{4.16}$$

Also, we generally denote

$$\frac{\delta}{\delta\bar{\alpha}} = -\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial}{\partial\bar{\alpha}_{,i}}\right) \qquad \text{and} \qquad \frac{\delta}{\delta\bar{\eta}} = \frac{\partial}{\partial\bar{\eta}}.$$

Finally, we conclude that *the Hamilton equations for thermoelasticity are given by the equations*

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}t} = -\frac{\delta H}{\delta X_L} \qquad \frac{\mathrm{d}\bar{\eta}}{\mathrm{d}t} = -\frac{\delta H}{\delta\bar{\alpha}} \qquad \frac{\mathrm{d}X_L}{\mathrm{d}t} = \frac{\delta H}{\delta\pi_L} \qquad \frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}t} = \frac{\delta H}{\delta\bar{\eta}} \qquad \forall t \in [t_1, t_2]. \tag{4.17}$$

The last two of equations (4.17) are a simple rewriting of equations (4.16). It is interesting to focus on the first two of the Hamilton equations (4.17). Using equations (4.14), one can prove that equations $(4.17)_{1,2}$ provide the pseudomomentum equation for thermoelasticity (3.20) and the entropy equation (3.25).

The first of equations (4.17) gives

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}t} = -\left[\frac{\partial H}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial X_{L,i}}\right)\right] \quad\Rightarrow\quad \frac{\partial\Lambda}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial X_{L,i}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial X_{L,t}}\right) = 0 \tag{4.18}$$

which is the equation of pseudomomentum (see equations (3.7) and (3.20)). The second one provides the entropy equation

$$\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}t} = \left[\frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial\bar{\alpha}_{,i}}\right)\right] \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial\bar{\alpha}_{,i}}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial\bar{\alpha}_{,t}}\right) = 0. \tag{4.19}$$

4.3. The case of classical thermoelasticity

We remark that what we have obtained up to this point concerns dissipation-less thermoelasticity (see the constitutive variables in relation $(2.4)_4$). We now use the previous analysis to explore the classical thermoelasticity. In this case, the free-energy function will be of the form

$$\Psi = \Psi\left(X_L, x_{i,L}, \frac{\partial\alpha}{\partial t}\right) \qquad \bar{\Psi} = \bar{\Psi}\left(X_L, X_{L,i}, \frac{\partial\bar{\alpha}}{\partial t}, \frac{\partial\bar{\alpha}}{\partial x_i}\right).$$

Note that the spatial derivative of $\bar{\alpha}$ automatically appears in the argument of $\bar{\Psi}$ due to relation (2.12). We now examine the Hamilton equation $(4.17)_1$:

$$\frac{\mathrm{d}\pi_L}{\mathrm{d}t} = -\left[\frac{\partial H}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial H}{\partial X_{L,i}}\right)\right] \quad\Rightarrow\quad \frac{\partial\Lambda}{\partial X_L} - \frac{\mathrm{d}}{\mathrm{d}x_i}\left(\frac{\partial\Lambda}{\partial X_{L,i}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\Lambda}{\partial X_{L,t}}\right) = 0.$$

At first sight, the above equation looks similar to equation (4.18), but we have to keep in mind that, for the case under study, the argument of Ψ is different. We calculate again every term in the above equation:

$$\frac{\partial\Lambda}{\partial X_{L,i}} = \frac{\partial\bar{K}}{\partial X_{L,i}} - \frac{\partial\bar{\Psi}}{\partial X_{L,i}} = J_F{}^{-1}\left(-x_{i,M}b^{\mathrm{mech}}_{ML} + \frac{\partial x_i}{\partial t}P_L\right) \tag{4.20}$$

$$\frac{\partial \Lambda}{\partial \left(\frac{\partial X_L}{\partial t}\right)} = \frac{\partial \bar{K}}{\partial \left(\frac{\partial X_L}{\partial t}\right)} - \frac{\partial \bar{\Psi}}{\partial \left(\frac{\partial X_L}{\partial t}\right)} = J_F{}^{-1} P_L \tag{4.21}$$

where

$$b_{ML}^{\text{mech}} = -(\delta_{ML} L + T_{Mj} x_{j,L}).$$

Inserting now equations (4.20) and (4.21) into equation (4.18), we obtain

$$-\frac{\mathrm{d}b_{ML}^{\text{mech}}}{\mathrm{d}X_M} + \frac{\partial P_L^{\text{mech}}}{\partial t} = \frac{\partial L}{\partial X_L} + \eta \frac{\partial \theta}{\partial X_L} \tag{4.22}$$

where

$$P_L^{\text{mech}} = -\rho \frac{\partial x_i}{\partial t} x_{i,L} = P_L - \eta \beta_L.$$

Equation (4.22) is the pseudomomentum equation for classical thermoelasticity, also obtained in Dascalu and Maugin (1995) following a completely different procedure.

To sum up we have formulated a Hamilton formulation for thermoelasticity given by equations (4.17). Also, we have proved that this formulation, when the constitutive relations (2.14) are adopted, provides the pseudomomentum equation and the entropy balance equation of the thermoelasticity of Green and Naghdi. Furthermore, the same formulation provides the pseudomomentum equation of classical thermoelasticity.

5. Poisson brackets and balance laws

We furthermore proceed to the concept of Poisson brackets in the framework of the proposed Hamiltonian structure. We will examine simultaneously elasticity and thermoelasticity, considering the indices A, B to run from 1 to 4 under the following convention:

$$X_A = (X_1, X_2, X_3, \bar{\alpha}) \qquad \pi_A = (\pi_1, \pi_2, \pi_3, \bar{\eta}).$$

Thus, we can write for any function f defined on the phase space

$$\mathcal{F}[X_L, \bar{\alpha}, \pi_L, \bar{\eta}] = \mathcal{F}[X_A, \pi_A] = \int_\Omega f(X_A, \pi_A)\, \mathrm{d}x \qquad L = 1, 2, 3 \quad A = 1, 2, 3, 4$$

where Ω is any regular subset of B_t.

Let f and g be two functions defined in the phase space (the space of canonical momenta π_L, $\bar{\eta}$ and positions X_L, $\bar{\alpha}$). The Poisson brackets are defined to be[8]

$$\{f, g\} = \int_\Omega \left[\frac{\delta f}{\delta X_A} \frac{\delta g}{\delta \pi_A} - \frac{\delta g}{\delta X_A} \frac{\delta f}{\delta \pi_A} \right] \mathrm{d}x. \tag{5.1}$$

Using definition (5.1) and equations (4.11), one can easily prove the following standard relations for a Hamiltonian structure:

$$\{f, X_A\} = -\int_\Omega \frac{\delta f}{\delta \pi_A}\, \mathrm{d}x \qquad \{f, \pi_A\} = -\int_\Omega \frac{\delta f}{\delta X_A}\, \mathrm{d}x \qquad A, B = 1, 2, 3, 4.$$

Furthermore, one can note that the Poisson brackets define a differentiation with respect to time. Indeed, if the Hamilton equations

$$\frac{\mathrm{d}\pi_A}{\mathrm{d}t} = -\frac{\delta H}{\delta X_A} \qquad \frac{\mathrm{d}X_A}{\mathrm{d}t} = \frac{\delta H}{\delta P_A} \tag{5.2}$$

hold and f is a function on the phase space, then we can write

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \{f, H\} \qquad t \in [t_1, t_2]. \tag{5.3}$$

[8] This definition has been proposed for elasticity by Maugin (1993).

The right part of equation (5.3) can be written in virtue of equations (5.2),

$$
\begin{aligned}
\{f, H\} &= \int_\Omega \left[\frac{\delta f}{\delta X_A} \frac{\delta H}{\delta \pi_A} - \frac{\delta H}{\delta X_A} \frac{\delta f}{\delta \pi_A} \right] \mathrm{d}x \\
&= \int_\Omega \left[\frac{\partial f}{\partial X_A} \frac{\mathrm{d}X_A}{\mathrm{d}t} + \frac{\mathrm{d}\pi_A}{\mathrm{d}t} \frac{\partial f}{\partial \pi_A} \right] \mathrm{d}x \\
&= \int_\Omega \left[\frac{\partial f}{\partial X_A} \frac{\partial X_A}{\partial t} + \frac{\partial f}{\partial \pi_A} \frac{\partial \pi_A}{\partial t} \right] \mathrm{d}x.
\end{aligned}
$$

In other words, the Poisson brackets define a differentiation in the phase space along the orbit described by the Hamilton equations.

It is more interesting to examine equation (5.3) setting the canonical momentum π_A and the Hamiltonian H at the position of f; namely if the Hamilton equations hold, we can write

$$
\frac{\mathrm{d}\mathcal{P}_A}{\mathrm{d}t} = \{\pi_A, H\} \qquad \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \{H, H\} \qquad t \in [t_1, t_2] \tag{5.4}
$$

where

$$
\mathcal{P}_A = \int_\Omega \pi_A(x_i, t)\, \mathrm{d}x \qquad \mathcal{H} = \int_\Omega H(x_i, t)\, \mathrm{d}x. \tag{5.5}
$$

Let us explore the latter formulation. Starting with equation $(5.4)_2$, we immediately show that

$$
\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega H(x_i, t)\, \mathrm{d}x = 0 \tag{5.6}
$$

which expresses the conservation of energy.

As concerns equation $(5.4)_1$, we write for its right side

$$
\begin{aligned}
\{\pi_A, H\} &= \int_\Omega \left[\frac{\delta \pi_A}{\delta X_B} \frac{\delta H}{\delta \pi_B} - \frac{\delta H}{\delta X_B} \frac{\delta \pi_A}{\delta \pi_B} \right] \mathrm{d}x = \int_\Omega -\frac{\delta H}{\delta X_A}\, \mathrm{d}x \\
&= \int_\Omega - \left[\frac{\partial H}{\partial X_A} - \frac{\mathrm{d}}{\mathrm{d}x_i} \left(\frac{\partial H}{\partial X_{A,i}} \right) \right] \mathrm{d}x \\
&= \int_\Omega -\frac{\partial H}{\partial X_A}\, \mathrm{d}x + \int_{\partial\Omega} \frac{\partial H}{\partial X_{A,i}} \nu_i\, \mathrm{d}x \\
&= \int_\Omega \frac{\partial \Lambda}{\partial X_A}\, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial \Lambda}{\partial X_{A,i}} \nu_i\, \mathrm{d}x.
\end{aligned}
$$

Hence, equation $(5.4)_1$ is written in the form

$$
\frac{\mathrm{d}\mathcal{P}_A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \pi_A(x_i, t)\, \mathrm{d}x = \int_\Omega \frac{\partial \Lambda}{\partial X_A}\, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial \Lambda}{\partial X_{A,i}} \nu_i\, \mathrm{d}x. \tag{5.7}
$$

Equation (5.7) can be separated into two equations; the first for the values of the index A from 1 to 3 and the second for the value 4. So, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \pi_L(x_i, t)\, \mathrm{d}x = \int_\Omega \frac{\partial \Lambda}{\partial X_L}\, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial \Lambda}{\partial X_{L,i}} \nu_i\, \mathrm{d}x \qquad L = 1, 2, 3 \tag{5.8}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \pi_4(x_i, t)\, \mathrm{d}x = \int_\Omega \frac{\partial \Lambda}{\partial X_4}\, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial \Lambda}{\partial X_{4,i}} \nu_i\, \mathrm{d}x \\
\Rightarrow\ &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \bar{\eta}(x_i, t)\, \mathrm{d}x = \int_\Omega \frac{\partial \Lambda}{\partial \bar{\alpha}}\, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial \Lambda}{\partial \bar{\alpha}_{,i}} \nu_i\, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial \bar{\Psi}}{\partial \bar{\alpha}_{,i}} \nu_i\, \mathrm{d}x \\
\Rightarrow\ &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \bar{\eta}(x_i, t)\, \mathrm{d}x = \int_{\partial\Omega} \frac{\partial \bar{\Psi}}{\partial \bar{\alpha}_{,i}} \nu_i\, \mathrm{d}x.
\end{aligned} \tag{5.9}
$$

Following Maugin (1993), we can view equation (5.8) as a balance law for the total pseudomomentum. On the left side we have the rate of pseudomomentum of the body, while on the right side we have the sources of pseudomomentum due to the inhomogeneities (the first term on the rhs) and the flow of pseudomomentum through the boundary of the body $\partial\Omega$. Thus, the total pseudomomentum is conserved if and only if the body is homogeneous and there is no flux across $\partial\Omega$.

Equation (5.9) represents the balance of entropy. If the right-hand term, which describes the flux of entropy, vanishes the total entropy of the body is conserved. Thus, no internal entropy sources appear, a result in accordance with the theory of dissipation-less thermoelasticity of Green and Naghdi upon which we based the present analysis.

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